1

Iterative Conditional Modes (ICM) :-

1. Assign observations $x_i \xleftarrow{\text{(recons.)}} y_i$

2. Iterate over pixels one-by-one and assign to each pixel as follows:

(a) Calculate the energy fn. at current i^{th} pixel:

$$E(\bar{x},\bar{y})\Big|_{\text{for case } x_i=+1} = h\sum_i x_i - \beta \underbrace{\sum_{\{i,j\}}}_{\text{neighbourhood clique}} x_i x_j - \eta \underbrace{\sum_i}_{\text{clique}} x_i y_i$$

clique | or — ; neighbourhood clique ┼ or ├ or ┤ or ┬ or

~~(b) Assign $x_n =$~~

and also $E(\bar{x},\bar{y})\Big|_{\text{for case } x_i=-1}$.

(b) $$x_i = \begin{cases} +1, & \text{if } E(\bar{x},\bar{y})\big|_{x_i=+1} < E(\bar{x},\bar{y})\big|_{x_i=-1} \\ -1 & \text{"} \quad \text{"} \quad > E(\bar{x},\bar{y})\big|_{x_i=-1} \end{cases}$$

Correlation is automatically decided by choosing direction of looping.

* What would be the no. of bits in this case?

$$E(\bar{x},\bar{y})\Big|_{x_i=-a} - E(\bar{x},\bar{y})\Big|_{x_i=+a}$$

$$= \Big(h\sum_{j\neq i} x_j + h x_i\Big|_{x_i=-a} - \beta\sum_{j\in ne(i)} x_j \cdot x_i\Big|_{x_i=-a} - \beta\sum_{\substack{j\in ne(k),\\ k\neq i}} x_j \cdot x_k$$

$$- \eta\sum_{j\neq i} x_i y_i - \eta\, x_i\Big|_{x_i=-a}\sum_j y_j\Big)$$

$$-\Big(h\sum_{j\neq i} x_j + h x_i\Big|_{x_i=+a} - \beta\sum_{j\in ne(i)} x_j \cdot x_i\Big|_{x_i=+a}$$

$$- \beta\sum_{\substack{j,k,\\ j\in ne(k),\\ k\neq i}} x_j \cdot x_k - \eta\sum_{j\neq i} x_i y_i - \eta\, x_i\Big|_{x_i=+a}\sum_j y_j\Big)$$

$$= \Big(h(-2a) - \beta(-2a)\sum_{j\in ne(i)} x_j - \eta(-2a)\sum_{\substack{\cancel{j\in ne(i)}\\ j}} y_j\Big)$$

(doesn't depend on $-a$ or $+a$, only on nodes local to x_i).

1) (a), (b), (c)

(only pairwise cliques) $\{(1-2), (2-3), \ldots, (N-1, N)\}$.

$$p(\bar{x}) = \frac{1}{Z} \Psi_{1,2}(x_1, x_2) \ldots \Psi_{N-1}(x_{N-1}, x_N)$$

(joint distribution obtained as a product of potentials)

$$p(x_n) = \sum_{x_1} \sum_{x_2} \cdots \sum_{x_{n-1}} \sum_{x_{n+1}} \cdots \sum_{x_N} p(\bar{x})$$

[ask what is the computational complexity]

(replace summations with integrations for Gaussians)

$$= \frac{1}{Z} \sum_{x_1} \sum_{x_2} \cdots \sum_{x_{n-1}} \sum_{x_{n+1}} \cdots \sum_{x_N} \Psi_{1,2}(x_1, x_2) \ldots \Psi_{N-1}(x_{N-1}, x_N)$$

$$= \left[\frac{1}{Z} \overbrace{\sum_{x_{n-1}} \Psi_{n-1}(x_{n-1}, x_n) \cdots \sum_{x_2} \Psi_{2,3}(x_2, x_3) \sum_{x_1} \Psi_{1,2}(x_1, x_2)}^{\text{fn. of } x_n}\right] \times$$

[now what is the computational complexity?]

$$\underbrace{\left[\sum_{x_{n+1}} \Psi_{n}(x_n, x_{n+1}) \sum_{x_{n+2}} \cdots \sum_{x_{N-1}} \Psi_{N-2}(x_{N-2}, x_{N-1}) \sum_{x_N} \Psi_{N-1}(x_{N-1}, x_N)\right]}_{(\text{fn. of } x_n)}$$

Defining: $\mu_\alpha(x_n) = \sum_{x_{n-1}} \Psi_{n-1}(x_{n-1}, x_n) \cdots \sum_{x_1} \Psi_{1,2}(x_1, x_2)$

$$= \boxed{\sum_{x_{n-1}} \Psi_{n-1}(x_{n-1}, x_n)\, \mu_\alpha(x_{n-1})}$$

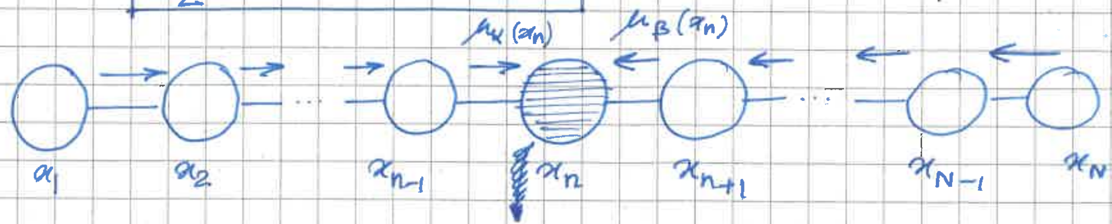
(forward message passing)

and $\boxed{\mu_\beta(x_n) = \sum_{x_{n+1}} \Psi_n(x_n, x_{n+1})\, \mu_\beta(x_{n+1})}$

(reverse/backward message passing)

Combining these two definitions,

$$p(x_n) = \boxed{\frac{1}{Z} \mu_\alpha(x_n) \cdot \mu_\beta(x_n)}$$



(d) $p(x_{n-1}, x_n) = \sum_{x_1} \sum_{x_2} \cdots \sum_{x_{n-2}} \sum_{x_{n+1}} \cdots \sum_{x_N} p(\bar{x})$

(d)

$$= \sum_{x_1} \sum_{x_2} \cdots \sum_{x_{n-2}} \sum_{x_{n+1}} \cdots \sum_{x_N} \frac{1}{Z} \Psi_{1,2}(x_1, x_2) \cdots \Psi_{n-1,n}(x_{n-1}, x_n) \Psi_n(x_n, x_{n+1}) \cdots \Psi_{N-1}(x_{N-1}, x_N)$$

$$= \overbrace{\left[\sum_{x_{n-2}} \Psi_{n-2}(x_{n-2}, x_{n-1}) \sum_{x_{n-3}} \Psi_{n-3}(x_{n-2}, x_{n-3}) \cdots \sum_{x_1} \Psi_{1,2}(x_1, x_2) \right]}^{= \mu_\alpha(x_{n-1})} \cdot \frac{1}{Z} \times$$

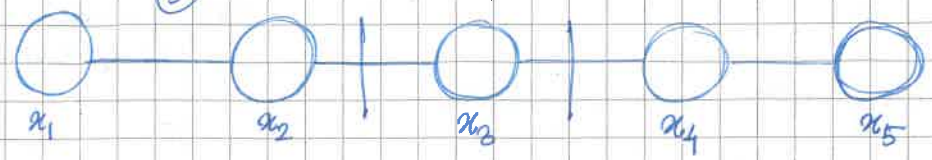
$$\Psi_{n-1,n}(x_{n-1}, x_n) \times$$

$$\underbrace{\left[\sum_{x_{n+1}} \Psi_n(x_n, x_{n+1}) \sum_{x_{n+2}} \Psi_{n+1}(x_{n+1}, x_{n+2}) \cdots \sum_{x_N} \Psi_{N-1}(x_{N-1}, x_N) \right]}_{= \mu_\beta(x_n)}$$

$$= \boxed{\frac{1}{Z} \mu_\alpha(x_{n-1}) \Psi_{n-1,n}(x_{n-1}, x_n) \mu_\beta(x_n)}$$

8/7

③



$$p(x_2 \mid x_3, x_5) = \frac{p(x_2, x_3, x_5)}{p(x_3, x_5)}$$

$x_2 \perp\!\!\!\perp x_5 \mid x_3$ (d-separation)

$$= \frac{p(x_2 \mid x_3)\, p(x_5 \mid x_3)\, p(x_3)}{p(x_3) \cdot p(x_5 \mid x_3)} = p(x_2 \mid x_3)$$

So to find $p(x_2 \mid x_3) = \dfrac{p(x_2, x_3)}{p(x_3)}$

$$p(x_3) = \frac{1}{Z} \mu_\alpha(x_3)\, \mu_\beta(x_3), \quad \text{(message passing)}$$

$$p(x_3, x_2) = \frac{1}{Z} \mu_\alpha(x_2) \cdot \psi_{2,3}(x_2, x_3)\, \mu_\beta(x_3) \quad \text{(message passing for two variables)}$$

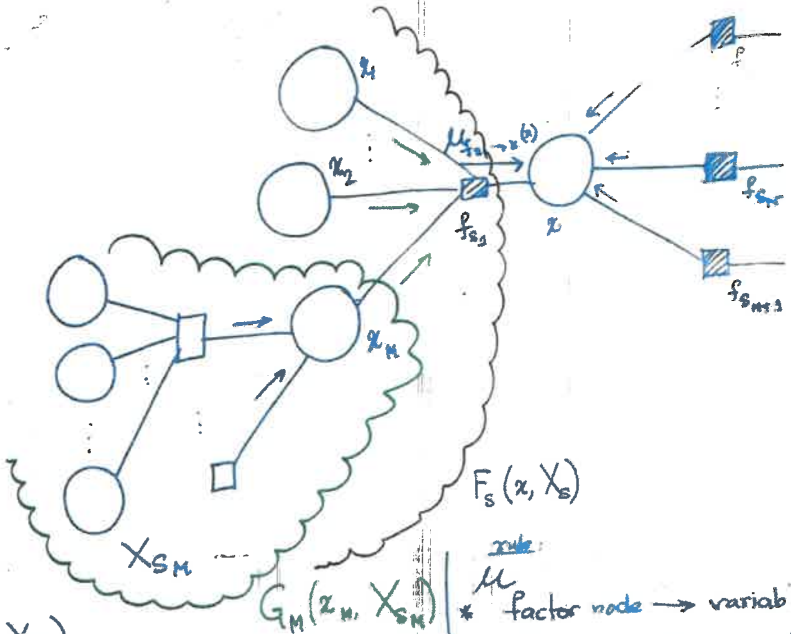
$$\therefore p(x_2 \mid x_3) = \boxed{\frac{\mu_\alpha(x_2)\, \psi_{2,3}(x_2, x_3)}{\mu_\alpha(x_3)}}$$

$$p(\bar{x}) = \prod_{s \in ne(\bar{x})} f_s(\bar{x}_s)$$
$$= \prod_{s \in ne(x)} F_s(x, X_s)$$

Calculating the marginal $p(x)$ means:

$$* \quad p(x) = \sum_{\bar{x} \setminus \{x\}} \prod_{s \in ne(x)} F_s(x, X_s)$$
$$= \prod_{s \in ne(x)} \sum_{\bar{x} \setminus \{x\}} F_s(x, X_s)$$
$$= \prod_{s \in ne(x)} \mu_{f_s \to x}(x)$$

(product of all messages from fac. nodes connected to x)



$$* \quad \mu_{f_s \to x}(x) = \sum_{X_s} F_s(x, X_s)$$
$$= \sum_{(x_1, x_2, \dots, x_M)} \underline{f_s(x, x_1, \dots, x_M) \cdot G_1(x_1, X_{s_1}) \dots G_M(x_M, X_{s_M})}$$

[black + green blanket]

$$= \sum_{(x_1, \dots, x_M)} f_s(x, x_1, \dots, x_M) \prod_{m \in ne(f_s) \setminus \{x\}} \sum_{X_{s_m}} G_m(x_m, X_{s_m})$$
$$= \sum_{(x_1, \dots, x_M)} f_s(x, x_1, \dots, x_M) \prod_{m \in ne(f_s) \setminus \{x\}} \mu_{x_m \to f_s}(x_m)$$

$$* \quad \mu_{x_m \to f_s}(x_m) = \sum_{X_{s_m}} G_m(x_m, X_{s_m})$$
$$= \sum_{X_{s_m}} \prod_{l \in ne(x_m) \setminus \{f_s\}} f_l(x_m, X_{m_l})$$
$$= \prod_{l \in ne(x_m) \setminus \{f_s\}} \mu_{f_l \to x_m}(x_m)$$

Initialize:

$\mu_{var. \to fac.} = \underline{1}$

$\mu_{fac. \to var.} = \underline{f(var.)}$

rule:

* $\mu_{\text{factor node} \to \text{variable node}}$ = weighted sum of product of incoming messages at factor node
* $\mu_{\text{variable node} \to \text{factor node}}$ = product of incoming messages at variable node